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Maximization of 3 variables function on unit sphere

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Let a, b, c be positive real numbers. Find

$$M := \max\left\{ayz + bzx + cxy \mid x, y, z \in \mathbb{R} \text{ and } x^2 + y^2 + z^2 = 1\right\}.$$

Solution. In case $a = b = c$ the problem becomes trivial because

$$\sum\left(x^2 - yz\right) = \frac{1}{2}\sum(x-y)^2 \geq 0$$

and, therefore, $M = 1$, then for further, due symmetry, assuming $a < b$ we will consider the following problem:
Find maximal value of $t \in \mathbb{R}^+$ such that inequality

$$x^2 + y^2 + z^2 \geq 2t\left(ayz + bzx + cxy\right) \iff$$

$$\iff x^2 + y^2 + z^2 - 2t\left(ayz + bzx + cxy\right) \geq 0 \tag{1}$$

holds for any $x, y, z \in \mathbb{R}$. Let

$$F(x, y, z, t) := x^2 + y^2 + z^2 - 2t\left(ayz + bzx + cxy\right).$$

Then by completing perfect square with respect to z we obtain

$$F(x, y, z, t) =$$

$$= \left(z - t\left(ay + bx\right)\right)^2 + \left(1 - a^2t^2\right)y^2 - 2yxt\left(c + abt\right) + \left(1 - b^2t^2\right)x^2$$

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Since

$$h(x,y,t) :=$$

$$= F(x,y,t(ay+bx),t) = \left(1-a^2t^2\right)y^2 - 2yxt(c+abt) + \left(1-b^2t^2\right)x^2 \geq 0$$

then $t < \frac{1}{a}$. Indeed, if $t = \frac{1}{a}$ then

$$h\left(x,0,\frac{1}{a}\right) = \left(1-\frac{b^2}{a^2}\right)x^2 < 0$$

for any $x \neq 0$ because $b > a$ and that is contradiction. If $t > \frac{1}{a}$ then $h(x,y,t) < 0$ for any $x,y > 0$.

Hence $t < \frac{1}{a} \iff 1-a^2t^2 > 0$ and, therefore, $h(x,y,t) \geq 0$ for any $x,y > 0$ iff

$$D(t) := t^2(c+abt)^2 - \left(1-a^2t^2\right)\left(1-b^2t^2\right) =$$

$$= 2abct^3 + \left(a^2+b^2+c^2\right)t^2 - 1 \leq 0$$

that is iff $t \leq t_*$ where t_* is unique positive root of equation

$$P(t,a,b,c) := 2abct^3 + \left(a^2+b^2+c^2\right)t^2 - 1 = 0$$

(because $D(0) = -1, D\left(\frac{1}{\sqrt[3]{2abc}}\right) > 0$ and $D(t)$ strictly increases on $[0,\infty)$).

Thus, t_* is maximal value of $t \in \mathbb{R}^+$ such that inequality **(1)** holds for any $x,y,z \in \mathbb{R}$ and for x,y,z such that $x^2+y^2+z^2 = 1$ we obtain inequality

$$ayz + bzx + cxy \leq \frac{1}{2t_*}. \tag{2}$$

Since

$$h(x,y,t) = \left(1-a^2t^2\right)\left(y - \frac{\left(abxt^2+cxt\right)}{1-a^2t^2}\right)^2 +$$

$$+\frac{2abct^3 + \left(a^2+b^2+c^2\right)t^2 - 1}{1-a^2t^2}\cdot x^2$$

then

$$F(x,y,z,t_*) =$$

$$= (z - t_*(ay+bx))^2 + (1-a^2t_*^2)\left(y - \frac{abt_*^2+ct_*}{1-a^2t_*^2}\cdot x\right)^2 \geq 0$$

with equality iff

$$y = \frac{abt_*^2+ct_*}{1-a^2t_*^2}\cdot x,\ z =$$

$$= t_*(ay+bx) = t_*\left(a\cdot\frac{abt_*^2+ct_*}{1-a^2t_*^2}\cdot x + bx\right) = \frac{bt_*+act_*^2}{1-a^2t_*^2}\cdot x$$

where x can be obtained from equation

$$x^2+y^2+z^2 = 1 \iff x^2\left(1+\left(\frac{abt_*^2+ct_*}{1-a^2t_*^2}\right)^2+\left(\frac{bt_*+act_*^2}{1-a^2t_*^2}\right)^2\right) = 1 \iff$$

$$\iff x^2\left(\left(1-a^2t_*^2\right)^2+\left(abt_*^2+ct_*\right)^2+\left(bt_*+act_*^2\right)^2\right) = \left(1-a^2t_*^2\right)^2 \iff$$

$$x^2\left(1-a^2t_*^2\right)\left(3-t_*^2\left(a^2+b^2+c^2\right)\right) = \left(1-a^2t_*^2\right)^2 \iff 2x^2\left(1+abct_*\right) =$$

$$= 1-a^2t_*^2 \iff x^2 = \frac{1-a^2t_*^2}{2\left(1+abct_*\right)}.$$

Thus, $M = \dfrac{1}{2t_*}$ as attainable upper bound for $ayz+bzx+cxy$.
As application of obtained result we consider the following problems:

Problem 1. Let A,B,C be angles of some non obtuse triangle. Find

$$\max\left\{yz\cos A + zx\cos B + xy\cos C \mid x,y,z\in\mathbb{R} \text{ and } x^2+y^2+z^2 = 1\right\}.$$

Solution. Let $(a,b,c) = (\cos A, \cos B, \cos C)$. Since

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$$

then

$$2abct^3 + \left(a^2 + b^2 + c^2\right) t^2 - 1 = 2abct^3 + \left(a^2 + b^2 + c^2\right) t^2 -$$

$$- \left(2abc + a^2 + b^2 + c^2\right) = 2abc\left(t^3 - 1\right) + \left(a^2 + b^2 + c^2\right)\left(t^2 - 1\right) =$$

$$= (t-1)\left(2abc\left(t^2 + t + 1\right) + \left(a^2 + b^2 + c^2\right)(t+1)\right)$$

and, therefore,

$$2abct^3 + \left(a^2 + b^2 + c^2\right) t^2 - 1 \le 0 \iff t \le 1$$

(because for any $t > 0$ holds $2abc\left(t^2 + t + 1\right) + \left(a^2 + b^2 + c^2\right)(t+1) > 0$).
Hence, $t_* = 1$ and $yz\cos A + zx\cos B + xy\cos C \le \dfrac{1}{2}$ with condition of equality $\begin{cases} z = \dfrac{b + ac}{1 - a^2} \cdot x \\ y = \dfrac{ab + c}{1 - a^2} \cdot x \\ x^2 + y^2 + z^2 = 1 \end{cases}$,where $a = \cos A, b = \cos B, c = \cos C$.

We have

$$x^2 + y^2 + z^2 = 1 \iff x^2\left(1 + \left(\frac{b + ac}{1 - a^2}\right)^2 + \left(\frac{ab + c}{1 - a^2}\right)^2\right) = 1 \iff$$

$$\Leftrightarrow \frac{4abc + b^2 + c^2 - 2a^2 + a^2\left(a^2 + b^2 + c^2\right) + 1}{\left(1 - a^2\right)^2} x^2 = 1 \iff \frac{2(abc + 1)}{1 - a^2} x^2 =$$

$$= 1 \Leftrightarrow x^2 = \frac{1 - a^2}{2(abc + 1)}$$

because

$$4abc + b^2 + c^2 - 2a^2 + a^2\left(a^2 + b^2 + c^2\right) + 1 =$$

$$= 4abc + 1 - 2abc - 3a^2 + a^2(1 - 2abc) + 1 = 2\left(1 - a^2\right)(abc + 1).$$

Hence,

$$z^2 = \left(\frac{b + ac}{1 - a^2}\right)^2 \cdot x^2 = \frac{(ac + b)^2}{2\left(1 - a^2\right)(abc + 1)},$$

$$y^2 = \left(\frac{ab+c}{1-a^2}\right)^2 \cdot x^2 = \frac{(ab+c)^2}{2(1-a^2)(abc+1)}.$$

Thus,

$$\max\left\{yz\cos A + zx\cos B + xy\cos C \mid x,y,z \in \mathbb{R} \text{ and } x^2+y^2+z^2=1\right\} = \frac{1}{2}$$

that is

$$yz\cos A + zx\cos B + xy\cos C \le \frac{1}{2}.$$

Problem 2. Find

$$M := \max\left\{yz + zx + 2xy \mid x,y,z \in \mathbb{R} \text{ and } x^2+y^2+z^2=1\right\}$$

Solution 1. Since $P(t,1,1,2) = 4t^3 + 6t^2 - 1$ and

$$4t^3 + 6t^2 - 1 \le 0 \iff (2t+1)\left(2t^2+2t-1\right) \le 0 \overset{t>0}{\iff} t \le \frac{\sqrt{3}-1}{2}$$

then $t_* = \dfrac{\sqrt{3}-1}{2}$ and $yz + zx + 2xy \le \dfrac{1}{2t_*} = \dfrac{\sqrt{3}+1}{2}$.

Solution 2. (direct, using completing square procedure)

$$F(x,y,z,t) := x^2+y^2+z^2 - 2t(yz+zx+2xy) =$$

$$= z^2 - 2t(x+y)z + \left(x^2 - 4txy + y^2\right) =$$

$$= (z - t(x+y))^2 + \left(x^2 - 4txy + y^2\right) - t^2(x+y)^2 =$$

$$= (z - t(x+y))^2 + x^2\left(1-t^2\right) - 2txy(t+2) + y^2\left(1-t^2\right)$$

Since

$$F(x,y,x+y,1) = -6xy < 0$$

if $xy > 0$ and for $t > 1$ we have

$$F(x,y,x+y,t) = x^2\left(1-t^2\right) - 2txy(t+2) + y^2\left(1-t^2\right) < 0$$

if $xy > 0$ then $t \in (0,1)$ and $x^2(1-t^2) - 2txy(t+2) + y^2(1-t^2) \geq 0$ for any x, y iff

$$t^2(t+2)^2 - (1-t^2)^2 \leq 0 \iff (2t+1)(2t^2+2t-1) \leq 0 \iff t \leq \frac{\sqrt{3}-1}{2}.$$

Thus, $\tau := \max t = \dfrac{\sqrt{3}-1}{2}$ and

$$yz + zx + 2xy \leq \frac{1}{2\tau}(x^2+y^2+z^2) = \frac{1}{2\tau} = \frac{\sqrt{3}+1}{2}$$

$$x^2(1-\tau^2) - 2\tau xy(\tau+2) + y^2(1-\tau^2) = x^2\left(1 - \left(\frac{\sqrt{3}-1}{2}\right)^2\right) -$$

$$-2\left(\frac{\sqrt{3}-1}{2}\right)xy\left(\frac{\sqrt{3}-1}{2}+2\right) + y^2\left(1-\left(\frac{\sqrt{3}-1}{2}\right)^2\right) = \frac{\sqrt{3}}{2}(x-y)^2.$$

Hence, $F(x, y, x+y, \tau) = 0$ iff $x = y$, $z = \dfrac{\sqrt{3}-1}{2} \cdot (x+y) = (\sqrt{3}-1)x$ and $x^2 + y^2 + z^2 = 1$,that is iff

$$2x^2 + (\sqrt{3}-1)^2 x^2 = 2\sqrt{3}(\sqrt{3}-1)x^2 = 1 \iff x^2 = \frac{1}{2\sqrt{3}(\sqrt{3}-1)} = \frac{3+\sqrt{3}}{12}$$

and, therefore,

$$x^2 = y^2 = \frac{3+\sqrt{3}}{12},\ z^2 = (\sqrt{3}-1)^2 \cdot \frac{3+\sqrt{3}}{12} = \frac{3-\sqrt{3}}{6}.$$

For $x = y = \sqrt{\dfrac{3+\sqrt{3}}{12}}, z = \sqrt{\dfrac{3-\sqrt{3}}{6}}$ we obtain

$$yz + zx + 2xy = \sqrt{\frac{3+\sqrt{3}}{12}} \cdot \sqrt{\frac{3-\sqrt{3}}{6}} + 2 \cdot \frac{3+\sqrt{3}}{12} =$$

$$= \frac{1}{3}\sqrt{3} + \frac{3+\sqrt{3}}{6} = \frac{\sqrt{3}+1}{2}.$$

Solution 3.(Using inequalities)

$$yz + zx + 2xy =$$

$$= z(y+x) + 2xy \le z\sqrt{2(y^2+x^2)} + y^2 + x^2 = z\sqrt{2(1-z^2)} + 1 - z^2.$$

Let $z := \cos t, t \in [0, \pi/2]$. Then

$$z\sqrt{2(1-z^2)} + 1 - z^2 = \sqrt{2}\cos t \sin t + \sin^2 t = \frac{1}{\sqrt{2}}\sin 2t + \frac{1}{2} -$$

$$-\frac{\cos 2t}{2} = \frac{1}{2} + \frac{1}{2}\left(\sqrt{2}\sin 2t - \cos 2t\right) \le \frac{1}{2} + \frac{1}{2}\cdot\sqrt{3}\sin(2t-\varphi) \le \frac{\sqrt{3}+1}{2}$$

(where $\varphi = \arcsin\frac{1}{\sqrt{3}}$). Since $\sin(2t-\varphi) = 1 \iff t = \frac{\pi}{4} + \frac{\varphi}{2}$ then equality in inequality

$$yz + zx + 2xy \le \frac{\sqrt{3}+1}{2}$$

occurs if* $z = \cos\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) = \sqrt{\frac{3-\sqrt{3}}{6}}$, and $x = y = \sqrt{\frac{1}{12}\sqrt{3} + \frac{1}{4}}$ because

$$2x^2 = 1 - \left(\sqrt{\frac{3-\sqrt{3}}{6}}\right)^2 = \frac{\sqrt{3}}{6} + \frac{1}{2} \iff x = \sqrt{\frac{1}{2}\left(\frac{\sqrt{3}}{6} + \frac{1}{2}\right)} = \sqrt{\frac{1}{12}\sqrt{3} + \frac{1}{4}}$$

$$*\cos\frac{\varphi}{2} = \sqrt{\frac{1+\cos\varphi}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{\sqrt{3}}}{2}} = \sqrt{\frac{3+\sqrt{6}}{6}},$$

$$\sin\frac{\varphi}{2} = \sqrt{\frac{1-\cos\varphi}{2}} = \sqrt{\frac{1-\frac{\sqrt{2}}{\sqrt{3}}}{2}} = \sqrt{\frac{3-\sqrt{6}}{6}},$$

$$\cos\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{3+\sqrt{6}}{6}} - \sqrt{\frac{3-\sqrt{6}}{6}}\right) = \sqrt{\frac{3-\sqrt{3}}{6}}.$$

Indeed,

$$\left(\sqrt{\frac{3+\sqrt{6}}{6}}-\sqrt{\frac{3-\sqrt{6}}{6}}\right)^2=\frac{3+\sqrt{6}}{6}+\frac{3-\sqrt{6}}{6}-2\sqrt{\frac{9-6}{36}}=\frac{3-\sqrt{3}}{3}$$

and, therefore,

$$\cos\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)=\frac{1}{\sqrt{2}}\sqrt{\frac{3-\sqrt{3}}{3}}=\sqrt{\frac{3-\sqrt{3}}{6}}.$$

Solution 4.(without trig) Consecutively using inequality $a+b\le\sqrt{2\left(a^2+b^2\right)}$ and Cauchy inequality we obtain

$$yz+zx+2xy=z\left(y+x\right)+2xy\le z\sqrt{2\left(y^2+x^2\right)}+y^2+x^2=$$

$$=z\sqrt{2\left(1-z^2\right)}+1-z^2=\sqrt{2}\sqrt{z^2-z^4}+\left(\frac{1}{2}-z^2\right)+\frac{1}{2}\le$$

$$\le\sqrt{\left(\sqrt{2}\right)^2+1^2}\cdot\sqrt{\left(\sqrt{z^2-z^4}\right)^2+\left(\frac{1}{2}-z^2\right)^2}+\frac{1}{2}=$$

$$=\sqrt{3}\cdot\sqrt{\left(\sqrt{z^2-z^4}\right)^2+\left(\frac{1}{2}-z^2\right)^2}+\frac{1}{2}=\frac{1+\sqrt{3}}{2}$$

with equality iff $x=y=\pm\sqrt{1-z^2}$ and

$$\frac{\sqrt{z^2-z^4}}{\sqrt{2}}=\frac{1}{2}-z^2\iff z=\pm\sqrt{\frac{3-\sqrt{3}}{6}},$$

Remark. *By the same way we can find*

$$\max\left\{a\left(yz+zx\right)+2bxy\mid x,y,z\in\mathbb{R}\text{ and }x^2+y^2+z^2=1\right\}.$$

Indeed, we have

$$a\left(yz+zx\right)+2bxy=az\left(y+x\right)+2bxy\le az\sqrt{2\left(y^2+x^2\right)}+b\left(y^2+x^2\right)=$$

$$=az\sqrt{2\left(1-z^2\right)}+b\left(1-z^2\right).$$

Let $z := \cos t, t \in [0, \pi/2]$. Then

$$az\sqrt{2(1-z^2)} + b(1-z^2) = a\sqrt{2}\cos t \sin t + b\sin^2 t = \frac{a}{\sqrt{2}}\sin 2t +$$

$$+\frac{b}{2} - \frac{b\cos 2t}{2} = \frac{b}{2} + \left(\frac{a}{\sqrt{2}}\sin 2t - \frac{b}{2}\cos 2t\right) \leq \frac{b}{2} + \sqrt{\frac{a^2}{2} + \frac{b^2}{4}} =$$

$$= \frac{b + \sqrt{2a^2 + b^2}}{2}.$$

Thus,

$$\max\left\{a(yz + zx) + 2bxy \mid x, y, z \in \mathbb{R} \text{ and } x^2 + y^2 + z^2 = 1\right\} = \frac{b + \sqrt{2a^2 + b^2}}{2}$$

Or, without using trigonometry, but applying Cauchy inequality:

$$az\sqrt{2(1-z^2)} + b(1-z^2) = b\left(\frac{a\sqrt{2}}{b} \cdot \sqrt{z^2 - z^4} + 1 \cdot \left(\frac{1}{2} - z^2\right) + \frac{1}{2}\right) \leq$$

$$\leq b\left(\sqrt{\left(\frac{a\sqrt{2}}{b}\right)^2 + 1^2} \cdot \sqrt{\left(\sqrt{z^2 - z^4}\right)^2 + \left(\frac{1}{2} - z^2\right)^2} + \frac{1}{2}\right) =$$

$$= b\left(\sqrt{\frac{2a^2 + b^2}{b^2}} \cdot \frac{1}{2} + \frac{1}{2}\right) = \frac{b + \sqrt{2a^2 + b^2}}{2}$$

with equality iff $x = y = \pm\sqrt{1 - z^2}$ and

$$\frac{\sqrt{z^2 - z^4}}{\frac{a\sqrt{2}}{b}} = \frac{1}{2} - z^2 \iff (2a^2 + b^2)z^4 - (2a^2 + b^2)z^2 + \frac{1}{2}a^2 = 0 \Leftrightarrow$$

$$\iff z^2 = \frac{2a^2 + b^2 + b\sqrt{2a^2 + b^2}}{2(2a^2 + b^2)} = \frac{b + \sqrt{2a^2 + b^2}}{2\sqrt{2a^2 + b^2}}.$$

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